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New uncertainties in QCD-QED rescaling factors using Quadrature Method

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Abstract

In this paper we briefly outline the quadrature method for estimating uncertainties in a function which depends on several variables, and apply it to estimate the numerical uncertainties in QCD-QED rescaling factors. We employ here the one-loop order in QED and three-loop order in QCD evolution equations of the fermion mass renormalization. Our present calculation is found to be new and also reliable compared to the earlier values employed by various authors.

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1 Introduction

An important aspect of particle physics is the concept of mass of a fermion. The experimental values of the fermion masses are the input values that we always take in various calculations, and they are generally known as physical masses of the fermions. Mathematically such physical mass is the running mass $m_f(\mu)$ of the fermion defined at the scale equal to its own physical mass, i.e., $m_f(\mu = m_f) = m_f(m_f)$. This is true for heavier quarks $f = t, b, c$ which have $m_f(m_f) > 1\text{GeV}$ and for charged leptons (τ, μ, e). However for lighter quarks ($f = s, d, u$) which have masses lesser than 1GeV , the physical mass is defined as $m_f(1\text{GeV})$. On the other hand, the fermion masses predicted in Grand unified theories (GUTs)[1], are defined at the grand unification scale M_X . In order to express these masses at lower energy scales we need the renormalisation of fermion masses from high scale to low scale[2]. At low energy scale below the top-quark mass scale m_t , the fermion mass renormalisation is governed by the QCD-QED symmetry gauge group $SU(3)_C \times U(1)_{em}$. Thus the fermion mass renormalisation from top-quark mass scale down to the physical quark mass scale is parametrised by the QCD-QED rescaling factors η_f defined by $\eta_f = m_f(m_f)/m_f(m_t)$. This is an important parameter which appears in many expressions related to the predictions of GUTs at low energy scale. The calculation of η_f involves the input values of e.m and strong gauge coupling constants α and α_3 . The uncertainties associated with the experimental values of α and α_3 will propagate through several intermediate scales upto low energies in the definition of QCD-QED rescaling factors, making the uncertainties larger and larger. In this respect, a careful estimation of uncertainties in η_f using a reliable method which can control the magnifying tendency of the uncertainties in η_f , is highly desirable. The uncertainties in η_f so far reported in the literature[4,12], do not match with each other, and the methods adopted by them are also not clearly specified. In this context we find the quadrature method[3] for the estimation of uncertainties of a function which depends on several variables, to be quite satisfactory and appropriate. It will be an important numerical exercise to estimate the uncertainties in η_f using the quadrature method and compare the results with the earlier values. We shall employ the one-loop order in QED and three-loop order in QCD evolution equations of the fermion mass renormalization.

The paper is organised as follows. In Section 2, we outline the procedure for the quadrature method. Section 3 is devoted to a brief note on how to

define QCD-QED rescaling factors and related quantities[4,5,6]. In Section 4, we present the numerical calculations of rescaling factors and their respective uncertainties. The paper concludes with a summary in Section 5.

2 Estimation of uncertainties in Quadrature Method

We define the quadrature method in the following way. If a function F depends on several variables x_i , where $i=1,2,\dots,n$, then the uncertainties in F resulting from the uncertainties of the independent variables x_i can be estimated by the following expression,

$$\delta F = \pm \sqrt{\sum \left(\frac{\partial F}{\partial x_i} \right)^2 (\delta x_i)^2} \quad (1)$$

where $\pm \left(\frac{\partial F}{\partial x_i} \right) (\delta x_i)$ is the error in F due to the error in i^{th} variable x_i . Here the uncertainties (or random errors of independent variables) add in quadrature[3] like orthogonal vectors, \vec{a} and \vec{b} whose resultant magnitude is given by $\sqrt{a^2 + b^2}$.

We give here three simple properties which will be needed to estimate the uncertainties in different quantities leading to the calculations of rescaling factors and also other parameters which depend on it (e.g., neutrino masses and mixings):

(i): If $F(x, y) = f(x, y) \times g(x, y)$, the uncertainties in F due to the random errors $\pm \delta x$ and $\pm \delta y$ in the independent variables x and y respectively, are given by

$$\delta F = \pm \sqrt{\left(g \frac{\partial f}{\partial x} + f \frac{\partial g}{\partial x} \right)^2 (\delta x)^2 + \left(g \frac{\partial f}{\partial y} + f \frac{\partial g}{\partial y} \right)^2 (\delta y)^2} \quad (2)$$

(ii): If $F(x, y) = \frac{f(x,y)}{g(x,y)}$, the uncertainties in F due to the random errors $\pm \delta x$ and $\pm \delta y$ in the independent variables x and y respectively, are given by

$$\delta F = \pm \sqrt{\left(\frac{1}{g} \frac{\partial f}{\partial x} - \frac{f}{g^2} \frac{\partial g}{\partial x} \right)^2 (\delta x)^2 + \left(\frac{1}{g} \frac{\partial f}{\partial y} - \frac{f}{g^2} \frac{\partial g}{\partial y} \right)^2 (\delta y)^2} \quad (3)$$

(iii): If $F(x, y) = \left(\frac{f(x)}{g(y)}\right)^p$, then the uncertainties in F due to the uncertainties $\pm\delta x$ and $\pm\delta y$ in the independent variables x and y respectively, are

$$\delta F = \pm F p \sqrt{\frac{1}{f^2} \left(\frac{\partial f}{\partial x}\right)^2 (\delta x)^2 + \frac{1}{g^2} \left(\frac{\partial g}{\partial y}\right)^2 (\delta y)^2} \quad (4)$$

3 QCD-QED rescaling factors

The QCD-QED rescaling factor η_f of the fermion f , which can take care of the fermion-mass-renormalisation from the top-quark mass scale down to the physical fermion mass scale $m_f(m_f)$, is defined as [7]

$$\eta_f = \frac{m_f(m_f)}{m_f(m_t)} = \left[\frac{m_f(m_f)}{m_f(m_t)} \right]_{QED} \times \left[\frac{m_f(m_f)}{m_f(m_t)} \right]_{QCD} \quad (5)$$

for f = b, c quarks having $m_f(m_f) > 1$ GeV,

$$\eta_f = \frac{m_f(1GeV)}{m_f(m_t)} = \left[\frac{m_f(1GeV)}{m_f(m_t)} \right]_{QED} \times \left[\frac{m_f(1GeV)}{m_f(m_t)} \right]_{QCD} \quad (6)$$

for f = s, d, u quarks having $m_f(m_f) < 1$ GeV, and

$$\eta_f = \left[\frac{m_f(m_f)}{m_f(m_t)} \right]_{QED}, f = \tau, \mu, e \quad (7)$$

Here $m_f(m_t)$ is the running mass of the fermion f at the top-quark mass scale, $\mu = m_t$, and $m_f(m_f)$ corresponds to the physical mass of the fermion. For convenience we also define a quantity $R^f(\mu, \mu')$ which represents the inverse of the QCD-QED rescaling factor in the narrow range, $\mu' - \mu$, ($\mu > \mu'$),

$$\begin{aligned} R^f(\mu, \mu') &= \left[\frac{m_f(\mu)}{m_f(\mu')} \right]_{QED} \times \left[\frac{m_f(\mu)}{m_f(\mu')} \right]_{QCD} \\ &= R_{QED}^f(\mu, \mu') \times R_{QCD}^f(\mu, \mu') \end{aligned} \quad (8)$$

The definitions of η_f and R^f in Eqs. (5)-(8), require only the contributions from the QED part for the charged leptons. Making use of Eq.(8) for successive narrow mass ranges: m_b-m_t , $m_\tau-m_b$, m_c-m_τ , $1\text{GeV}-m_c$, $m_s-1\text{GeV}$,

m_μ - m_s , m_d - m_μ , m_u - m_d , m_e - m_u , the QCD-QED rescaling factor η_f defined in Eqs.(5)-(7), can be rewritten as

$$\begin{aligned}\eta_b &= \eta_b(m_b, m_t) = 1/R^b(m_b, m_t), \\ \eta_\tau &= \eta_\tau(m_\tau, m_t) = 1/\left[R_{QED}^l(m_\tau, m_b)R_{QED}^l(m_b, m_t)\right], l = \tau, \mu, e, \\ \eta_c &= \eta_c(m_c, m_t) = 1/[R^c(m_c, m_\tau)R^c(m_\tau, m_b)R^c(m_b, m_t)], \\ \eta_{s,d} &= \eta_{s,d}(1GeV, m_t) = \eta_b/\left[R^{s,d}(1GeV, m_c)R^{s,d}(m_c, m_\tau)R^{s,d}(m_\tau, m_b)\right], \\ \eta_u &= \eta_u(1GeV, m_t) = \eta_c/[R^u(1GeV, m_c)], \\ \eta_\mu &= \eta_\mu(m_\mu, m_t) = \eta_\tau/\left[R_{QED}^l(m_\mu, m_s)R_{QED}^l(m_s, 1GeV)\right], \\ \eta_e &= \eta_e(m_e, m_t) = \eta_\mu/\left[R_{QED}^l(m_e, m_u)R_{QED}^l(m_u, m_d)R_{QED}^l(m_d, m_\mu)\right] \quad (9)\end{aligned}$$

We use one-loop order in QED and three-loop order in QCD evolution equations of fermion mass renormalisation for the evaluations of R_{QED}^f and R_{QCD}^f respectively. The contribution of one-loop QED, running from the scale μ' to μ , to the rescaling factors through Eq.(8) is now given by[7]

$$R_{QED}^f(\mu', \mu) = \left[\frac{\alpha(\mu)}{\alpha(\mu')}\right]^{r_f^{QED}}, \mu > \mu', \quad (10)$$

where

$$\begin{aligned}r_f^{QED}(\mu', \mu) &= \gamma_0^{QED}/b_0^{QED}, \\ \gamma_0^{QED} &= -3Q_f^2, \\ b_0^{QED} &= \frac{4}{3}\left[3\sum Q_u^2 + 3\sum Q_d^2 + \sum Q_e^2\right]. \quad (11)\end{aligned}$$

The summation in Eqs.(11) is over the active fermions at the relevant mass scale, and f is the specific fermion under consideration. We employ the one loop RGE for the estimation of e.m. gauge couplings $\alpha(\mu)$ at successive renormalisation points,

$$\frac{1}{\alpha(\mu')} = \frac{1}{\alpha(\mu)} + \frac{b_0^{QED}}{2\pi} \ln\left(\frac{\mu}{\mu'}\right) \quad (12)$$

The three-loop QCD running quark mass formula is given by [7,8]

$$m_q(\mu) = \hat{m}_q \left[b_0 \frac{\alpha_3(\mu)}{2\pi}\right]^{\gamma_0/b_0} \times \left(1 + A \frac{\alpha_3(\mu)}{4\pi} + B \left[\frac{\alpha_3(\mu)}{4\pi}\right]^2\right) \quad (13)$$

where \hat{m}_q is the scale-invariant quark mass, and

$$A = \frac{\gamma_1}{b_0} - \frac{b_1\gamma_0}{b_0^2},$$

$$B = \frac{1}{2} \left[A^2 + \frac{\gamma_2}{b_0} + \frac{b_1^2\gamma_0}{b_0^3} - \frac{b_1\gamma_1}{b_0^2} - \frac{b_2\gamma_0}{b_0^2} \right] \quad (14)$$

The QCD β -functions and anomalous dimensions are given as,

$$\gamma_0 = 4,$$

$$\gamma_1 = \frac{202}{3} - \frac{20}{9}n_f,$$

$$\gamma_2 = \frac{3747}{3} - \left(\frac{160}{3}\xi(3) + \frac{2216}{27} \right)n_f - \frac{140}{81}n_f^2,$$

$$b_0 = 11 - \frac{2}{3}n_f,$$

$$b_1 = 102 - \frac{38}{3}n_f,$$

$$b_2 = \frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2, \quad (15)$$

where n_f is the number of quark flavours at the relevant mass scale and $\xi(3) = 1.202$. The QCD rescaling contribution to R^f in the relevant mass-range $(\mu' - \mu)$ defined in Eq.(8), can be obtained from Eq.(13) as

$$R_{QCD}^f(\mu, \mu') = \left[\frac{\alpha_3(\mu)}{\alpha_3(\mu')} \right]^{\gamma_0/b_0} \times \frac{1 + A \frac{\alpha_3(\mu)}{4\pi} + B \left[\frac{\alpha_3(\mu)}{4\pi} \right]^2}{1 + A \frac{\alpha_3(\mu')}{4\pi} + B \left[\frac{\alpha_3(\mu')}{4\pi} \right]^2} \quad (16)$$

since $\alpha_3(\mu)$ changes smoothly with μ , so values of the constants A and B remain as same for a given energy range including the limits.

Using Eqs.(10) and (14) via (8), the QCD-QED rescaling factors can be estimated. The values of $\alpha_3(\mu)$ in Eq.(16) can be obtained by solving the three-loop QCD RGE with $(\alpha_3 = g_3^2/4\pi)$, [9]

$$\mu \frac{dg_3(\mu)}{d\mu} = \beta(g_3(\mu)) = -\frac{b_0}{16\pi^2}g_3^3 + \frac{b_1}{(16\pi^2)^2}g_3^5 - \frac{b_2}{(16\pi^2)^3}g_3^7 \quad (17)$$

The conventional solution of Eq.(17) having a constant of integration called QCD dimensional parameter Λ -which provides a parametrisation of the μ dependence of $\alpha_3(\mu)$ [9]is given by

$$\begin{aligned}\alpha_3(\mu) = \frac{4\pi}{\beta_0 \ln(\mu^2/\Lambda^2)} &[1 - \frac{2\beta_1}{\beta_0^2} \frac{\ln[\ln(\mu^2/\Lambda^2)]}{\ln(\mu^2/\Lambda^2)} + \frac{4\beta_1^2}{\beta_0^4 \ln^2(\mu^2/\Lambda^2)} \times \\ &\left(\left(\ln[\ln(\mu^2/\Lambda^2)] - \frac{1}{2} \right)^2 + \frac{\beta_2 \beta_0}{8\beta_1^2} - \frac{5}{4} \right)]\end{aligned}\quad (18)$$

where

$$\beta_0 = b_0, \beta_1 = b_1/2, \beta_2 = 2b_2. \quad (19)$$

Since β -function coefficients change by discrete amount as flavour thresholds are crossed while integrating the differential equation(17) for $\alpha_3(\mu)$, Λ also changes to take care of the validity of Eq.(18) for all values of μ . This leads to the concept of different $\Lambda^{(n_f)}$ for each range of μ corresponding to an effective number of quark flavours n_f . In the \bar{MS} scheme, one finds the relation among different $\Lambda^{(n_f)}$ [10] as

$$\begin{aligned}\Lambda_{MS}^{(4)} &\simeq \Lambda_{MS}^{(5)} \left[m_b / \Lambda_{MS}^{(5)} \right]^{2/25} \left[2 \ln \left[m_b / \Lambda_{MS}^{(5)} \right] \right]^{(963/14375)} \\ \Lambda_{MS}^{(4)} &\simeq \Lambda_{MS}^{(3)} \left[\Lambda_{MS}^{(3)} / m_c \right]^{2/25} \left[2 \ln \left[m_c / \Lambda_{MS}^{(3)} \right] \right]^{-(107/1875)}\end{aligned}\quad (20)$$

The QCD parameter $\Lambda_{MS}^{(5)}$ which corresponds to $n_f = 5$ can be calculated by using Eq.(18) with the input experimental value of $\alpha_3(M_Z)$ and then $\Lambda_{MS}^{(4)}$ and $\Lambda_{MS}^{(3)}$ can be obtained from Eq.(20) through a search programme. Subsequently, $\alpha_3(\mu)$ and R_{QCD}^f can be calculated for the energy, $\mu < M_Z$.

If we consider SUSY above M_Z , assuming the existence of one-light Higgs doublet (N_H) and five quark flavours ($n_f = 5$) in the energy range $m_t - M_Z$, the strong gauge coupling at the scale m_t are evaluated using RGE solution with one loop order

$$\frac{1}{\alpha_3(\mu)} = \frac{1}{\alpha_3(M_Z)} + \frac{3}{2\pi} \ln \frac{\mu}{M_Z} \quad (21)$$

for $\mu > M_Z$.

4 Numerical calculations and results:

For numerical calculation we take the input values of fermion masses as [9]

$$\begin{aligned} m_t &= 175 \text{GeV}, m_b = 4.25 \text{GeV}, m_\tau = 1.785 \text{GeV}, \\ m_c &= 1.25 \text{GeV}, m_s = 0.18 \text{GeV}, m_\mu = 0.105 \text{GeV}, \\ m_d &= 0.009 \text{GeV}, m_u = 0.005 \text{GeV}, m_e = .00051 \text{GeV}, \end{aligned} \quad (22)$$

and CERN-LEP data [11] of $\alpha^{-1}(M_Z) = 127.9 \pm 0.1$ and strong coupling constant $\alpha_3(M_Z) = 0.1172 \pm .002$ at $M_Z = 91.187 \text{GeV}$, which is referred to as Case-I. For comparison, we also estimate the uncertainties using values of $\alpha_3(M_Z) = 0.120 \pm .0028$ referred to as Case-II and $\alpha_3(M_Z) = .118 \pm .007$ made use for earlier calculation of uncertainties [12], now referred to as Case-III.

Now using above data, we evaluate the gauge couplings $\alpha(\mu)$ and $\alpha_3(\mu)$ at various renormalisation points starting from m_t down to the individual fermion mass (for quark it stops at 1GeV) from their corresponding Eqs.(12),(18)and (21) and values are presented in Table-1 and Table-5 respectively. The co-efficients of β -functions and anomalous dimensions in RGEs for QED and QCD with constants A, B relevant in different energy ranges are estimated and shown in Tables-1,2,3. We also calculate the inverse of the fermion mass renormalisation factors $R_f^{QED}(\mu, \mu')$ for charged leptons, up-quarks and down-quarks shown in Table-1,2 and $R_f^{QCD}(\mu, \mu')$ in three Cases-I, II, III in Table-5. The values of QCD dimensional parameters Λ_{MS}^f for estimating $\alpha_3(\mu)$ at various energy scales are presented in Table-4.

We present below the numerical values of QCD-QED rescaling factors η_f ($f=b, c, s, d, u$) with uncertainties for different values of $\alpha_3(M_Z)$. It is observed that the uncertainties in the input values of $\alpha_3(M_Z)$ propagate down to the lower energy scale, causing the uncertainties in η_f larger and larger:

Case	$\alpha_3(M_Z)$	η_b	η_c	$\eta_{s,d}$	η_u
I	$0.1172 \pm .002$	$1.5380^{+0.0361}_{-0.0335}$	$2.1785^{+0.2186}_{-0.1580}$	$2.4329^{+0.2607}_{-0.2182}$	$2.4489^{+0.2607}_{-0.2186}$
II	$0.1200 \pm .0028$	$1.5649^{+0.0503}_{-0.0479}$	$2.2487^{+0.2337}_{-0.2067}$	$2.6486^{+0.4491}_{-0.3620}$	$2.6660^{+0.4239}_{-0.3844}$
III	$0.1180 \pm .007$	$1.5458^{+0.1282}_{-0.1165}$	$2.2110^{+0.6298}_{-0.4548}$	$2.5106^{+1.1991}_{-0.7162}$	$2.5271^{+1.1991}_{-0.7162}$

The rescaling factors for charged leptons are found as

$\eta_\tau = 1.017 \pm .0007$, $\eta_\mu = 1.027 \pm .0038$, $\eta_e = 1.046 \pm .0099$ for all cases.

For Case-III with the value of $\alpha_3(M_Z) = .118 \pm .007$, the results estimated by Deshpande and Keith [12] using the same value of $\alpha_3(M_Z) = .118 \pm .007$, have small uncertainties as follows.

$$\eta_b = 1.53^{+0.07}_{-0.06}, \eta_c = 2.09^{+0.27}_{-0.19}, \eta_{s,d} = 2.36^{+0.53}_{-0.29}, \eta_u = 2.38^{+0.52}_{-0.30}.$$

These low uncertainties may be resulted from an apparent mistake in their estimation of uncertainties in

$$\alpha_3^{-1}(\mu_t) = 9.30^{+0.054}_{-0.047},$$

instead, it should be corrected as

$$\alpha_3^{-1}(\mu_t) = 9.30^{+0.54}_{-0.47}.$$

which will lead to larger uncertainties in η_f .

The present estimation of uncertainties in η_f using quadrature method is very convincing and also it regulates the propagation of uncertainties while running the energy scale from high to low scales. These uncertainties can reliably be used in other low energy calculations e.g., neutrino masses and mixings in see-saw mechanism in low energy scale[13].

5 Summary and conclusion:

To summarise, we have outlined the procedure for estimating the uncertainties using the quadrature method. In particular, we employ this method to

estimate the numerical uncertainties in QCD-QED rescaling factors η_f acquired from the uncertainties in input values of $\alpha_3(M_Z)$ and $\alpha(M_Z)$ while running the energy scale from high to low. We have used the three-loop order in QCD and one-loop order in QED evolution equations to calculate the rescaling factors while their uncertainties estimated by the quadrature method are found to be new in comparison to earlier estimates.

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Table-1: Coefficients of β -functions in the RGEs for QED, the values of r_f^{QED} in different energy ranges $(\mu - \mu')$, $\mu > \mu'$, the inverse of the gauge coupling $\alpha^{-1}(\mu)$ and inverse of the rescaling factors $R_f^{QED}(\mu, \mu')$ for charged leptons (e, μ, τ) with input value of $\alpha^{-1}(M_Z) = 127.9 \pm 0.1$.

Energy range $(\mu - \mu')$	b_0^{QED}	r_f^{QED} $(f = e, \mu, \tau)$	$\alpha^{-1}(\mu)$	R_f^{QED} $(f = e, \mu, \tau)$
$m_t - m_b$	80/9	-0.3375	126.98 ± 0.1	0.9865 ± 0.0002
$m_b - m_\tau$	79/9	-0.3553	132.24 ± 0.1	0.9968 ± 0.0005
$m_\tau - m_c$	64/9	-0.4219	133.40 ± 0.1	0.9987 ± 0.0006
$m_c - 1GeV$	16/3	-0.5625	133.81 ± 0.1	0.9990 ± 0.0008
$1GeV - m_s$	16/3	-0.5625	133.99 ± 0.1	0.9939 ± 0.0007
$m_s - m_\mu$	44/9	-0.6136	135.45 ± 0.1	0.9981 ± 0.0089
$m_\mu - m_d$	32/9	-0.8438	135.87 ± 0.1	0.9914 ± 0.0011
$m_d - m_u$	29/9	-0.9643	137.26 ± 0.1	0.9979 ± 0.0014
$m_u - m_e$	4/3	-2.2500	137.55 ± 0.1	0.9921 ± 0.0032
$m_e - 0$	138.04 ± 0.1	...

Table-2: Coefficients of β -functions in the RGEs for QED, the values of r_f^{QED} in different energy ranges $(\mu - \mu')$, $\mu > \mu'$, and inverse of the rescaling factors $R_f^{QED}(\mu, \mu')$ for up- and down-quarks with renormalisation scale stopping at 1GeV.

Energy scale $(\mu - \mu')$	b_0^{QED}	r_f^{QED} (f=u,c,t)	r_f^{QED} (f=s,d,b)	R_f^{QED} (f=u,ct)	R_f^{QED} (f=s,d,b)
$m_t - m_b$	80/9	-0.15	-0.0375	0.9839 ± 0.0002	0.9985 ± 0.0001
$m_b - m_\tau$	79/9	-0.1579	-0.0395	0.9986 ± 0.0002	0.9997 ± 0.0001
$m_\tau - m_c$	64/9	-0.1875	-0.0469	0.9994 ± 0.0003	0.9999 ± 0.0001
$m_c - 1GeV$	16/3	-0.25	-0.0625	0.9996 ± 0.0004	0.9999 ± 0.0001

Table-3: Coefficients of β functions in RGEs, anomalous dimensions for QCD and values of the constants A and B defined in Eq.(10) with $\gamma_0 = 4$ throughout the computation.

Energy scale $(\mu - \mu')$	γ_1	γ_2	b_0	b_1	b_2	A	B	n_f
$m_t - m_b$	506/9	474.89	23/3	116/3	9769/54	4.7020	24.0123	5
$m_b - m_\tau$	526/9	636.62	25/3	154/3	21943/54	4.0563	22.2280	4
$m_\tau - m_c$	526/9	636.62	25/3	154/3	21943/54	4.0563	22.2280	4
$m_c - 1GeV$	546/9	794.90	9	64	34767/54	3.5802	21.9434	3

Table-4: The values of QCD dimensional parameters Λ_{MS}^f with uncertainties estimated for three different values of $\alpha_3(\mu)$ for cases I, II, III.

case	$\Lambda_{MS}^{(5)} \text{GeV}$	$\Lambda_{MS}^{(4)} \text{GeV}$	$\Lambda_{MS}^{(3)} \text{GeV}$
I	$0.1995^{+0.0243}_{-0.0217}$	$0.2877^{+0.0313}_{-0.0283}$	$0.3375^{+0.0325}_{-0.0295}$
II	$0.2329^{+0.0364}_{-0.0334}$	$0.3305^{+0.0459}_{-0.0429}$	$0.3838^{+0.0472}_{-0.0448}$
III	$0.2087^{+0.0934}_{-0.0716}$	$0.2995^{+0.1177}_{-0.0942}$	$0.3498^{+0.1191}_{-0.1001}$

Table-5: The values of $\alpha_3(\mu)$ and $R_{QCD}^f(\mu - \mu')$ with uncertainties estimated for quarks using three-loop order in QCD evolution equations for three different input experimental values of $\alpha_3(M_Z)$. Here QM stands for the quadrature method employed here.

case / parameter	$m_t - m_b$	$m_b - m_\tau$	$m_\tau - m_c$	$m_c - 1GeV$	$1GeV - 0$
I / $\alpha_3(\mu)$	$0.1068^{+0.0017}_{-0.0016}$	$0.2225^{+0.0079}_{-0.0073}$	$0.3149^{+0.0170}_{-0.0152}$	$0.3967^{+0.0303}_{-0.0254}$	$0.4819^{+0.0507}_{-0.0400}$
(QM) / R_{QCD}^f	$0.6512^{+0.0153}_{-0.0142}$	$0.8185^{+0.0310}_{-0.0279}$	$0.8682^{+0.0503}_{-0.0430}$	$0.8900^{+0.0700}_{-0.0564}$	
II / $\alpha_3(\mu)$	$0.1091^{+0.0022}_{-0.0023}$	$0.2334^{+0.0112}_{-0.0107}$	$0.3382^{+0.0257}_{-0.0234}$	$0.4233^{+0.0466}_{-0.0389}$	$0.5568^{+0.1001}_{-0.0728}$
(QM) / R_{QCD}^f	$0.6400^{+0.0206}_{-0.0196}$	$0.8055^{+0.0430}_{-0.0396}$	$0.8696^{+0.0732}_{-0.0629}$	$0.8439^{+0.1129}_{-0.0851}$	
III / $\alpha_3(\mu)$	$0.1075^{+0.0057}_{-0.0059}$	$0.2257^{+0.0285}_{-0.0253}$	$0.3213^{+0.0663}_{-0.0508}$	$0.4079^{+0.1334}_{-0.0828}$	$0.5001^{+0.2652}_{-0.1254}$
(QM) / R_{QCD}^f	$0.6479^{+0.0538}_{-0.0489}$	$0.8152^{+0.1161}_{-0.0927}$	$0.8633^{+0.2030}_{-0.1348}$	$0.8753^{+0.3351}_{-0.1722}$	

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